Appl. Math. Lett. Vol. 1, No. 1, pp. 49-51, 1988
0893-9659/88 $3.00 + 0.00

A singular perturbation problem with boundary and turning point resonance.

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1. Introduction.

In this note we study the combined effects of an interior turning point and a singular boundary in a singular perturbed two point boundary value problem of second order on the interval (0, 1):

$$L_\varepsilon u \;:=\; -\,\varepsilon\,(x\,p(x,\varepsilon)\,u')' + x\,(x-a)\,q(x,\varepsilon)\,u' \;=\; \lambda\,u\,, \qquad ' := d/dx \tag{1.1}$$

with the boundary conditions

$$u(1)\,\sin(\phi) + u'(1)\,\cos(\phi) = 0, \tag{1.2}$$

where ε is a small positive parameter and $0 < a < 1$. The functions p and q are smooth and strictly positive. The boundary point $x = 0$ is a regular singular point for the differential equation (1.1). The interior point $x = a$ is a (second order) turning point. Both points are special for the singular perturbation problem, in that they both "generate" a set of eigenvalues for the BVP that converge to definite finite values as ε tends to zero.

This note is mainly motivated by a recent paper of Wazwaz & Hanson [3] dealing with the same problem. Their analysis of the eigenvalue problem (1.1-2) is based on uniform approximations of the solutions of the differential equation. This makes the asymptotics of the eigenvalues quite involved. E.g. in the case where two eigenvalues of both special points coalesce, they need a separate analysis of the asymptotics, and they suggest that something special is happening there. Moreover, the boundary condition they impose on u at $x = 0$ is quite unnatural, cf. [1]. At this regular singular point only one solution of the differential equation is bounded, hence the solution manifold needs no additional restriction at this point.

In this note we shall indicate, how by the spectral comparison technique, explained in [2], a much easier analysis can be made of the eigenvalue problem. We shall sketch a proof of the result:

Theorem: *Every element of the sets*

$$S := \{naq(0,0) \mid n = 1, 2, \cdots\} \quad \text{and} \quad T := \{naq(a,0) \mid n = 0, 1, \cdots\} \tag{1.3}$$

is the limit of an eigenvalue of (1.1-2) and every eigenvalue satisfies the estimate

$$\lambda(\varepsilon) = s + O(\varepsilon) \qquad (\varepsilon \to 0) \tag{1.4}$$

for some $s \in S \cup T$. If $s \in S \cap T$, it is the limit of two distinct eigenvalues.

We note that the differential equation (1.1) differs from the equation in [3] by a minus sign in order to make the eigenvalues positive.

2. The eigenvalue problem and its eigenvalues.

The problem (1.1-2) is selfadjoint with respect to the weighted inner product $(\cdot\,,\,\cdot)_w$,

$$(u\,,\,v)_w := \int_0^1 u(x)\,v(x)\,w(x,\varepsilon)\,dx\,, \qquad \|u\|_w := \sqrt{(u,u)_w}\,, \tag{2.1}$$

where the weighting function w is defined by

$$w(x,\varepsilon) := \exp(-\frac{Q(x,\varepsilon)}{\varepsilon})\,, \qquad Q(x,\varepsilon) := \int_a^x \frac{(\xi-a)q(\xi,\varepsilon)}{p(\xi,\varepsilon)}\,d\xi\,. \tag{2.2}$$

If we arrange its eigenvalues $\{\lambda_k \mid k = 0, 1, \cdots\}$ in an increasing sequence, they are characterized variationally by Rayleigh's quotient,

$$\lambda_k = \inf_{E\subset H,\ \dim E = k+1}\ \sup_{u\in E,\ u\neq 0} \frac{(L_\varepsilon\, u\,,\,u)_w}{(u\,,\,u)_w}\,, \tag{2.3}$$

where H is the domain of definition of the operator L_ε. An approximation of the eigenvalue λ_k is obtained by inserting for E a collection trial functions consisting of approximations of the first k+1 eigenfunctions, cf. [2, § 2].

In this case we use two sets of trial functions. The first one is obtained by stretching eq. (1.1) at the turning point by

$$\xi := (x-a)\sqrt{q_a / 2\varepsilon p_a}\,, \quad p_a := p(a,0)\,, \quad q_a := q(a,0)\,,$$

and solving the lowest order part. The only solutions for which Rayleigh's quotient does not explode for $\varepsilon \to 0$, are the Hermite polynomials,

$$\chi_n(x,\varepsilon) := H_n(\xi)\,, \quad n = 0, 1, \cdots\,. \tag{2.4}$$

occurring at the parameter values $\lambda = aq_a n$, $n = 0, 1, \cdots$. The second set is obtained analogously by the stretching of the equation at the singular boundary point

$$\zeta := aq_0 x / \varepsilon p_0\,, \quad p_0 := p(0,0)\,, \quad q_0 := q(0,0)\,.$$

This set consists of the Laguerre polynomials multiplied by a rapidly decaying exponential,

$$\psi_n(x,\varepsilon) := L_n(\zeta)\exp(-\zeta)\,, \tag{2.5}$$

which occur at the parameter values $\lambda = (n+1)aq_0$, $n = 0, 1, \cdots$.

Evaluation of the Rayleigh quotient for these trial function yields the desired eigenvalue approximations for $\varepsilon \to 0$ and $n = 0, 1, \cdots$:

$$\frac{(L_\varepsilon\chi_n\,,\,\chi_n)_w}{(\chi_n\,,\,\chi_n)_w} = aq_a n + O(\varepsilon)\,, \quad \frac{(L_\varepsilon\psi_n\,,\,\psi_n)_w}{(\psi_n\,,\,\psi_n)_w} = aq_0(n+1) + O(\varepsilon)\,. \tag{2.6}$$

Higher order approximations of the eigenvalues and eigenfunctions can be obtained by inserting in the stretched equation a formal power series for the eigenvalue in powers of ε at $x = 0$ and of $\varepsilon^{1/2}$ at $x = a$. Expansion results in a recursive system of equations in which the coefficients of the eigenvalue expansion are uniquely determined by the condition that Rayleigh's quotient of the solution should not explode, cf. [2, § 7].

3. The comparison problems.

In order to prove the theorem, we squeeze the spectrum of (1.1-2) between the known spectra of related operators using (2.3). To the operator L_ε is associated the bilinear form B_ε

$$B_\varepsilon(u,v) := (L_\varepsilon u,v)_w = \varepsilon\ (xpu'\ ,\ v')_w + \varepsilon \tan(\phi)\ p\ u\ v\ w\Big|_{x=1} . \tag{3.1}$$

The domain of definition of B_ε can be extended to the set H^1 of all locally square integrable functions u on (0,1), for which $||u||_w$ and $||x^{1/2}u'||_w$ are finite. (If $\phi = \pi/2 \bmod \pi$ in (1.2), the functions u should also satisfy $u(1) = 0$.) By varying Rayleigh's quotient in this space we obtain precisely the eigenvalues of (1.1-2).

By choosing the variations within the subspace of functions that are zero at some point α between 0 and a, the infimum (minimum) cannot decrease and we obtain an upper estimate of each eigenvalue. This choice effectively splits the eigenvalue problem (1.1-2) into problems on the subintervals $(0,\alpha)$ and $(\alpha,1)$, both with the additional boundary condition $u(\alpha) = 0$. Thus we separate completely the influences of the regular singularity at $x = 0$ and the turning point $x = a$ on the spectrum.

Enlarging the the space of variations by no longer requiring continuity of the trial functions at α (i.e. $x^{1/2}u'$ is square integrable only on the subintervals $(0,\alpha)$ and $(\alpha,1)$) we obtain a lower bound for each eigenvalue. This actually splits the eigenvalue problem in two problems on $(0,\alpha)$ and $(\alpha,1)$, both with the additional "free" boundary condition $u'(\alpha) = 0$.

So we obtain four comparison eigenvalue problems. For the ones on $(\alpha,1)$ we can apply immediately the results of [2]. For the problems on $(0,\alpha)$ we can show by the same means that the numbers naq_0 , $n = 1, 2, \cdots$, approximate the eigenvalues up to $O(\varepsilon)$, independently of the type of boundary condition imposed artificially at $x = \alpha$. All together this shows that the eigenvalues are within $O(\varepsilon)$ neighbourhoods of the comparison problem, and this establishes the theorem.

References.

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